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REPORT ON THE PHD THESIS OF KAROL BOLBOTOWSKI

This letter constitutes my report on the PhD thesis of Karol Bolbotowski. It is in fact an outstanding thesis – among the best I have seen in my 40 years on the faculty of the Courant Institute.

I'll start with some basic information. The thesis is entitled *Elastic Bodies and Structures of the Optimum Form, Material Distribution, and Anisotropy.* It was done under the supervision of Tomasz Lewinski and Piotr Rybka. It presents three distinct (but related) research thrusts, concerning:

- *Free material design* (Chapters 3 and 4),
- *Optimal design of planar membranes* (Chapter 5), and
- *Optimal design of forms* (Chapter 6).

Three research papers have already been completed based on this material: *Setting the free material design problem through the methods of optimal mass distribution* (coauthored with Tomasz Lewinski, available as arXiv:2004.11084); *Optimal design versus maximal Monge-Kantorovich metrics* (coauthored with Guy Bouchitté, to appear in Arch Rational Mech Anal, available as arXiv:2104.04894); and *Optimal vault problem – form finding through 2D convex program* (available as arXiv:2104.07148). These papers correspond to Chapters 3 & 4, Chapter 5, and Chapter 6 respectively.

Turning to the science, it is natural to start by briefly summarizing the scientific foundation upon which this thesis builds. It includes:

(i) The theory of *Michell trusses.* This theory links the design of maximally-rigid truss-like continua to a dual pair of convex optimimizations, one with L^1 growth and the other with L^∞ constraints. Due to the $L^1 - L^\infty$ character of this theory, the optimal structures it produces can concentrate on lower-dimensional sets.

(ii) The existing literature on *free material design.* This work assumes that the cost per unit area of a Hooke's law H is a convex function of H, and uses methods from PDE-constrained convex optimization to minimize the cost of a 2D or 3D structure subject to constraints on its mechanical performance. This work usually places upper and lower bounds on the Hooke's law, to ensure that the equations of elasticity are uniformly elliptic; as a result, concentrated structures of the type seen in Michell trusses do not arise.

(iii) A body of work on *optimal vaults*. This work designs shells that robustly support vertical loads (due e.g. to gravity) by ignoring resistance to bending and insisting that the stress be everywhere compressive (that is, negative semi-definite). While the concept is old, there was (prior to Bolbotowski's work) little vision about how such shells could be identified (let alone optimized) using methods from convex optimization.

Turning now to Bolbotowski's accomplishments:

Chapters 3 & 4 of the thesis develop a theory of free material design that places neither lower nor upper bounds on the Hooke's law, and that includes the Michell truss problem as a special case. Importantly: the constitutive laws permitted by this theory are much more general than linear elasticity; for example, one can assume that the material sustains only compressive stresses, or only tensile stresses. This extension of the theory of free material design is a timely and very natural development. While the mathematical toolkit required to implement it was largely familiar from the literature on Michell trusses, this work is nevertheless impressive for its elegance, efficiency, and generality.

Chapter 5 of the thesis studies the optimal design of a planar membrane when the loading is transverse. This is quite different from – but just as natural as – the 2D Michell truss problem (which considers in-plane loads and prohibits out-of-plane deformation). The analysis uses a von Kármán ansatz, which is entirely appropriate in the small-slope regime where the optimal designs are expected to be found. Since a membrane wrinkles to avoid compression, the theory assumes that the membrane can sustain only tensile stresses. Rather surprisingly, despite the apparent nonconvexity of the von Kármán framework, the restriction to tensile stresses restores convexity, thereby permitting an approach analogous to that of Chapters 3 & 4. Another surprise emerges later in the chapter: the convex optimizations that arise in this setting have close connections to the Monge-Kantorovich theory of optimal transport. I should emphasize that while the free material design framework of Chapters 3 & 4 plays a crucial role in Chapter 5, the presence of out-of-plane deformation makes the analysis fundamentally different from what Bolbotowski had done before.

Chapter 6 of the thesis studies the optimal design of load-bearing surfaces. Briefly: this chapter finds optimal surfaces by (i) considering a "convex relaxation" of the design problem, which can be solved by the methods of Chapter 5, then (ii) proving that under certain circumstances the relaxed problem is equivalent to the original one, in the sense that it does indeed determine an optimal surface. This two-step process builds on ideas introduced in the late 70's and early 80's by William Prager and George Rozvany; however Bolbotowski's work is very different from what came before, providing an elegant and rather general theory where Prager and Rozvany had previously presented what amount to a few examples.

Bolbotowski's work on free material design (Chapters 3 & 4) is a contribution of the type one expects to see in a PhD thesis: research that advances our understanding of an existing research frontier. His work on the optimal design of membranes and forms (Chapters 5 and 6) are, however, contributions of a type that one rarely sees in a PhD thesis: research that introduces an entirely new viewpoint, thereby identifying an entirely new research frontier. As usual in seminal contributions, this work answers important questions but leaves more work to be done. I therefore enjoyed reading the final sections of Chapters 5 and 6, which include thoughtful discussions of some of the questions that remain open.

My summary of the thesis has emphasized its theoretical elements. I note, however, that there are also many instructive (and impressive) numerical examples.

I have not yet commented on the exposition. It is excellent: both efficient and clear. I particularly liked the Introduction, which offers a very clear summary of the thesis' objectives, methodology, and accomplishments (Section 1.2) and a fine review of the relevant literature (Section 1.1).

No literature review is perfect, and I would like to mention one thread that Section 1.1 omits. Chapter 4 introduces, as a modeling hypothesis, the idea that a membrane can sustain only tensile stresses. This is supported (and its consequences have been studied) by a number of articles on the mechanics of thin elastic sheets. I would point especially to the paper *Rigorous derivation of Föppl's theory for clamped elastic membranes leads to relaxation* by S. Conti, F. Maggi, and S. Müller, SIAM J Math Anal 38(2), 2006, 657–680, which uses the same von Kármán framework that's the starting point of Bolbotowski's analysis. But it also seems worth noting analogous results are known using models that avoid the small-slope hypothesis implicit in the von Kármán framework; examples of such work include *Relaxed energy densities for large deformations of membranes*, IMA J Appl Math 52(3), 1994, 297–308 and *The nonlinear membrane model as variational limit of nonlinear three-dimensional elasticity* by H. Le Dret and A. Raoult, J Math Pures Appl 74, 1995, 549–578.

I have already noted that Bolbotowski's exposition is both efficient and clear. I would have preferred, however, that the small-slope hypothesis implicit in the use of the von Kármán framework be noted more explicitly. Actually, the first introduction of a small-slope hypothesis is in Section 1.1, where optimal arch-grids and cable-networks are motivated by a brief discussion of the "funicular problem." We are reminded that when bending resistance is ignored, the equilibrium of a cable with prestress s is modeled by $-su'' = f$, where f is the vertical force and u is the vertical displacement. This is correct – but of course it relies on a small-slope hypothesis, and I would have preferred that this be mentioned explicitly.

The preceding criticisms are of course very minor. Overall, Karol Bolbotowski's thesis is truly outstanding – achieving a level of excellence one sees only rarely, even at the best universities. In particular (as noted earlier), Chapters 5 and 6 introduce an entirely new viewpoint concerning the optimal design of membranes and load-bearing surfaces, thereby identifying an entirely new research frontier. With this feature in mind, I am pleased to recommend that this work be awarded whatever distinction your university reserves for its very best PhD theses.

Sincerely yours,

Robert V. Kohn
Silver Professor of Mathematics